Government of **Western Australia**
School Curriculum and Standards Authority

WACE

MATHEMATICS

3C/3D

Calculator-free

WACE Examination 2012

Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

When examiners design an examination, they develop provisional marking keys that can be reviewed at a marking key ratification meeting and modified as necessary in the light of candidate responses.

Section One: Calculator-free (50 Marks)

Question 1 (4 marks)

Let $f(x)=(x+3)(1-x^2)^5$.

The derivative of $f(x)$ can be written in the form $f'(x)=(1-x^2)^4(ax^2+bx+c)$.
Determine the values of a,b and c.

Solution
$f'(x)=1(1-x^2)^5+(x+3)(5)(1-x^2)^4(-2x)$ $=(1-x^2)^4\left[(1-x^2)-10x(x+3)\right]$ $=(1-x^2)^4(-11x^2-30x+1)$ $a=-11, b=-30, c=1$
Specific behaviours
✓ differentiates into the form $u'v+v'u$ ✓ determines u' and v' correctly ✓ correctly determines the remaining factor once $(1-x^2)^4$ is factorised out ✓ simplifies to obtain values of a,b and c.

Question 2 **(5 marks)**

A company made 16 motorbikes of three different types.

Each type A motorbike cost $5000 to make, while each type B motorbike cost $2000 and each type C cost $1000. The company spent $65 000 making the 16 motorbikes.

The number of type A motorbikes made was three times the total number of type B and C motorbikes.

Let a = number of type A motorbikes,
b = number of type B motorbikes, and
c = number of type C motorbikes.

Some of the information above is represented by the two equations:

$$a+b+c=16$$
$$5a+2b+c=65$$

(a) Write down a third equation which, together with the equations above, is sufficient to determine the values of a, b and c. (1 mark)

Solution
$a=3(b+c)=3b+3c$
Specific behaviours
✓ states correct equation

(b) How many of each type of motorbike were made? (4 marks)

Solution
Since $a=3(b+c)$ and $a+b+c=16$, $a=12$ and $b+c=4$. Then $60+2b+c=65$, so $2b+c=5$. From $b+c=4$ and $2b+c=5$ we have $b=1$, and so $c=3$. Alternatively: $a+b+c=16$ ...Eq1 $5a+2b+c=65$ ...Eq2 $a-3b-3c=0$ ...Eq3 $3b+4c=15$ ...$5\times\text{Eq1}-\text{Eq2}$ $4b+4c=16$ ...$\text{Eq1}-\text{Eq3}$ $b=1$ ...using last two equations $c=3$ ...back-substitution $a=12$ ...back-substitution
Specific behaviours
✓ determines that $a=12$ and $b+c=4$ using ratio ✓ substitutes to find $2b+c=5$ ✓ determines that $b=1$ ✓ determines that $c=3$ Alternatively: ✓ eliminates a variable from one equation correctly ✓ eliminates the same variable from another equation correctly ✓ solves for one of the remaining variables ✓ back-substitutes to solve for other variables

Question 3 **(7 marks)**

Let A, B, C, D, E, F and G be points on the graph of a continuous function $f(x)$.

The table below contains information about the sign of $f(x)$, $f'(x)$ and $f''(x)$ at these points.

Point	**A**	**B**	**C**	**D**	**E**	**F**	**G**
x	– 4	–3	–1	0	1	2	4
$f(x)$	+	0	–	0	+	+	+
$f'(x)$	–	–	0	+	+	0	+
$f''(x)$	+	+	+	0	–	0	+

There are no other points at which $f(x)$, $f'(x)$ or $f''(x)$ are equal to zero.

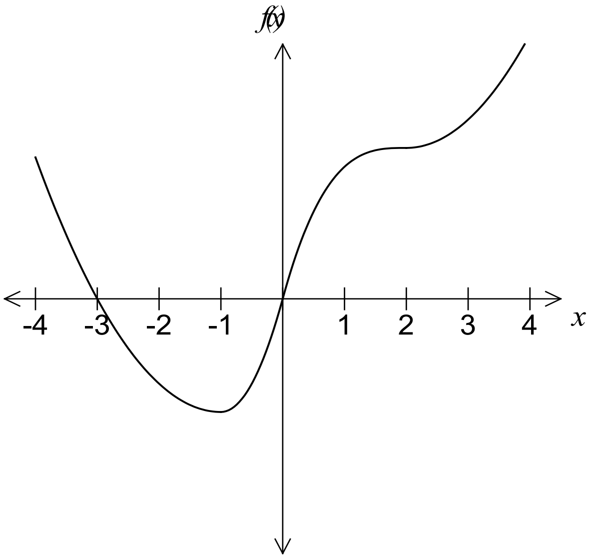
(a) Which point is a local minimum? (1 mark)

Solution
C
Specific behaviours
✓ identifies correct point

(b) Describe the nature of the graph at point F. (2 marks)

Solution
Horizontal point of inflection
Specific behaviours
✓ identifies that F is a point of inflection ✓ identifies that $f(x)$ is horizontal at F

(c) Sketch the function on the axes below. (4 marks)

Solution

Specific behaviours
✓ sketches local minimum at C and horizontal point of inflection at F, as per Part (a) ✓ sketches x-intercepts at –3 and 0 ✓ sketches a point of inflection at $x = 0$ ✓ completes graph correctly

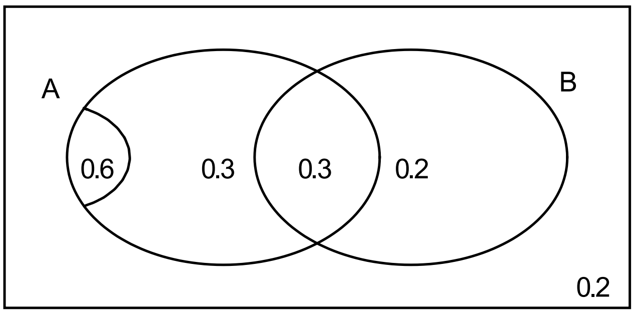
Question 4 **(7 marks)**

Two events A and B have the following properties.

P(A ∪ B) = 0.8

P(A ∩ B) = 0.3

P(A) = 0.6



(a) Calculate:

(i) P(B). (1 mark)

Solution
P(A ∪ B) = P(A) + P(B) – P(A ∩ B). So P(B) = 0.5
Specific behaviours
✓ determines the value of P(B)

(ii) $P(\overline{A} \cap B)$. (2 marks)

Solution
$P(B) = P(A \cap B) + P(\overline{A} \cap B)$ (or draw Venn diagram to show this) $0.5 = 0.3 + P(\overline{A} \cap B)$ $0.2 = P(\overline{A} \cap B)$
Specific behaviours
✓ identifies that $P(B) = P(A \cap B) + P(\overline{A} \cap B)$ through equation or diagram ✓ solves for $P(\overline{A} \cap B)$

(b) For a third event C, $P(C \mid B) = 0.4$.

(i) Calculate $P(B \cap C)$. (1 mark)

Solution
$P(B \cap C) = P(C \mid B)\,P(B) = 0.4 \times 0.5 = 0.2$
Specific behaviours
✓ determines the value of $P(B \cap C)$

(ii) If events B and C above are independent, and events A and C are mutually exclusive, determine the value of $P(A \cup C)$. (3 marks)

Solution
Since B and C are independent, $P(C) = P(C \mid B) = 0.4$ Since A and C are mutually exclusive, $P(A \cup C) = P(A) + P(C) = 1$
Specific behaviours
✓ determines the value of $P(C)$ ✓ determines the value of $P(A \cup C)$ ✓ uses independence and mutual exclusivity in calculations

Question 5 **(9 marks)**

(a) Evaluate $\int_0^1 8x\left(2x^2-1\right)^7 dx.$ (3 marks)

Solution
$\int_0^1 8x\left(2x^2-1\right)^7 dx = \left[\frac{\left(2x^2-1\right)^8}{4}\right]_0^1$ $= \frac{1}{4} - \frac{1}{4}$ $= 0$
Specific behaviours
✓ integrates to the form $k\left(2x^2-1\right)^8$ ✓ determines that $k = \frac{1}{4}$ ✓ evaluates the integral correctly

(b) If $\frac{dy}{dx} = \frac{2}{x^2} + 4x$, and $y = 3$ when $x = 2$, determine the value of y when $x = 5$. (3 marks)

Solution
$\frac{dy}{dx} = 2x^{-2} + 4x$ $y = -2x^{-1} + 2x^2 + c = \frac{-2}{x} + 2x^2 + c$ $3 = \frac{-2}{2} + 2\left(2^2\right) + c$ $-4 = c$ $y\big\|_{x=5} = \frac{-2}{5} + 2\left(5^2\right) - 4 = 45\frac{3}{5} = 45.6$
Specific behaviours
✓ integrates correctly   ✓ determines the value of the constant of integration   ✓ evaluates y correctly when $x = 5$

(c) Evaluate $\int_1^2 \frac{d}{dx}\left(\frac{x^3}{x^2+1}\right)dx$. (3 marks)

Solution
$\int_1^2 \frac{d}{dx}\left(\frac{x^3}{x^2+1}\right)dx = \left[\frac{x^3}{x^2+1}\right]_1^2$ $= \frac{8}{5} - \frac{1}{2}$ $= \frac{11}{10}$ $= 1.1$
Specific behaviours
✓ uses the Fundamental Theorem of Calculus   ✓ substitutes limits of integration correctly   ✓ simplifies correctly

Question 6 **(6 marks)**

(a) Express $\frac{5}{x+5}-\frac{2}{x+2}$ in the form $\frac{ax+b}{(x+5)(x+2)}$, where a and b are constants.

(2 marks)

Solution
$\frac{5}{x+5}-\frac{2}{x+2}=\frac{5(x+2)-2(x+5)}{(x+5)(x+2)}$ $=\frac{3x}{(x+5)(x+2)}$
Specific behaviours
✓ forms a single fraction with the correct denominator   ✓ simplifies numerator correctly

(b) Using your answer to Part (a) or otherwise, solve the inequality $\frac{5}{x+5}>\frac{2}{x+2}$.

(4 marks)

Solution

$$\frac{3x}{(x+5)(x+2)}>0$$

Critical points at $x=-5,-2,0$

x	$x<-5$	$-5<x<-2$	$-2<x<0$	$x>0$
$\frac{3x}{(x+5)(x+2)}$	–	+	–	+

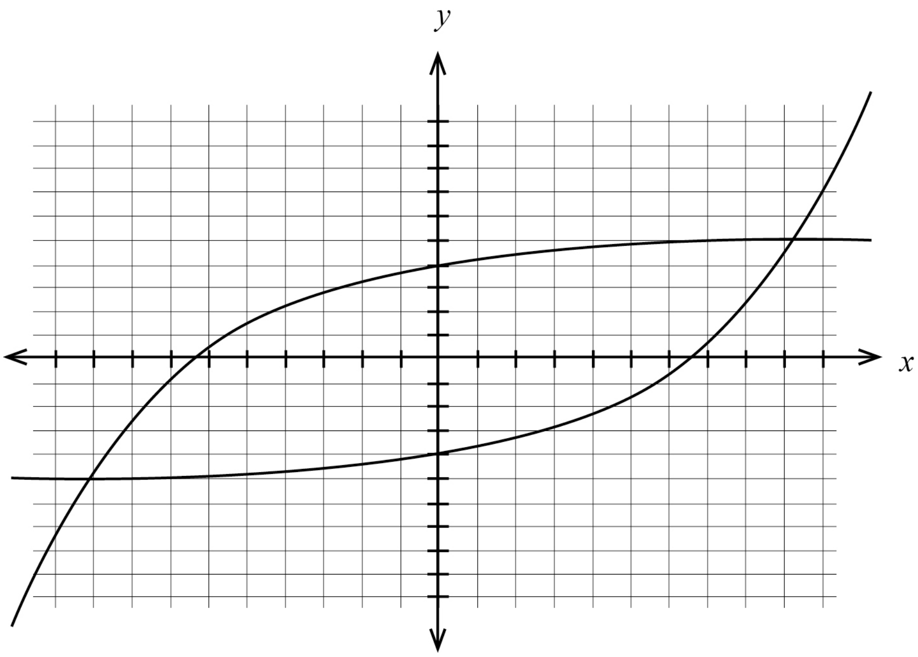
So $-5<x<-2$ or $x>0$. Alternative notation: $(-5,-2)\cup(0,\infty)$

Specific behaviours

✓ identifies all critical points
✓ specifies the interval $-5<x<-2$ as a solution
✓ specifies the interval $x>0$ as a solution
✓ includes no other interval as a solution

Question 7 **(6 marks)**

Part of the graph of $y = a + be^{cx}$, where a, b and c are constants, is shown below.



(a) Which of the constants a, b and c are positive, and which are negative? Justify your answers. (3 marks)

Solution
From the graph of $y = e^x$, this graph has been reflected in the y-axis, since it has a horizontal asymptote as $x \to \infty$, rather than as $x \to -\infty$. So $c < 0$. From the graph of $y = e^x$, this graph has also been reflected in the x-axis, since it tends to $-\infty$, rather than ∞, as $x \to -\infty$. So $b < 0$. As $x \to \infty$, $y \to a$. From the location of the asymptote, $a > 0$. So a is positive and b and c are both negative.
Specific behaviours
✓ states sign of a, with justification ✓ states sign of b, with justification ✓ states sign of c, with justification

(b) Sketch on the same axes the graph of $y = -a - be^{-cx}$. (3 marks)

Solution
Shown on axes above
Specific behaviours
✓ y-intercept located correctly ✓ x-intercept located correctly ✓ graph has horizontal asymptote in correct location

Question 8 **(6 marks)**

A continuous function $f(x)$ is increasing on the interval $0 < x < 3$ and decreasing on the interval $3 < x < 6$. Some of its values are given in the table below.

x	0	1	2	3	4	5	6
$f(x)$	5	16	27	32	25	0	– 49

The function $F(x)$ is defined, for $0 \le x \le 6$, by $F(x) = \int_0^x f(t)\,dt$.

(a) At which value of x in the interval $0 \le x \le 6$ is $F(x)$ greatest? Justify your answer. (2 marks)

Solution
$x = 5$ $F(x)$ can be interpreted as the signed area under the graph of $f(x)$ and to the right of $x = 0$. So long as $f(x) > 0$, this area will increase – which is true up until $x = 5$.
Specific behaviours
✓ determines correct value of x ✓ explains in terms of sign of $f(x)$

(b) At which value of x in the interval $0 \le x \le 6$ is $F'(x)$ greatest? Justify your answer. (2 marks)

Solution
$x = 3$ $F'(x) = f(x)$, and the maximum value of $f(x)$ occurs when $x = 3$.
Specific behaviours
✓ determines correct value of x ✓ explains in terms of the maximum value of $f(x)$

(c) Use the values of $f(x)$ in the table to show that $48 \le F(3) \le 75$. (2 marks)

Solution
The area under the graph of $f(x)$ between $x = 0$ and $x = 3$ is bounded by rectangles defined by $x = 0, 1, 2, 3$ and $y = f(0), f(1), f(2), f(3)$. The lower bounding rectangles have an area of $5 + 16 + 27 = 48$ The upper bounding rectangles have an area of $16 + 27 + 32 = 75$ So $48 \le F(3) \le 75$
Specific behaviours
✓ explains that the area corresponding to $F(3)$ is bounded by rectangles. ✓ shows the calculation of the lower and upper bounds determined by the rectangles.